# AD 273 996



UNCLASSIFIED

62-3-1

AFOSR/DRA-61-3

AFOSR/DRA-61-3



# AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
# TECHNICAL REPORT

PROBABILITY CONSIDERATIONS ON DESTROYING ICBM'S
WITH INTERCEPTOR SATELLITES

Herbert Knothe



## DIRECTORATE OF RESEARCH ANALYSIS
### HOLLOMAN AIR FORCE BASE
### NEW MEXICO

November 1961

AFOSR/DRA-61-3

AFOSR/DRA-61-3

# PROBABILITY CONSIDERATIONS ON DESTROYING ICBM's WITH INTERCEPTOR SATELLITES

by

Herbert Knothe

Mathematics Division
Directorate of Research Analysis

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
OFFICE OF AEROSPACE RESEARCH
UNITED STATES AIR FORCE
Holloman Air Force Base, New Mexico

November 1961

## ABSTRACT

This paper presents a method for calculating the probability that a certain minimum number of a set of ICBM's can be destroyed by a given number of interceptor satellites.

This report is approved for publication.

James H. Ritter

JAMES H. RITTER
Colonel, USAF
Director, Research Analysis Directorate

PROBABILITY CONSIDERATIONS ON DESTROYING ICBM's
WITH INTERCEPTOR SATELLITES

The problem, as defined by Dr. Fritz Hoehndorf, may be formulated as follows:

Let $m$ and $n$ be the numbers of interceptors and ICBM's, respectively. Each of the $m$ interceptors is supposed to attack exactly one of the $n$ ICBM's. One ICBM may be attacked by one or more interceptors. The probability that one interceptor destroys the attacked ICBM is $p$. The probability $P_{ik}$ that the $i^{th}$ interceptor attacks the $k^{th}$ ICBM shall be independent of $i$, $k$.

What is the probability that at least $n - \mu$ ICBM's are destroyed or, in other words, at most $\mu$ ICBM's escape destruction?

Let us first answer the question: What is the probability of destroying exactly $j$ ICBM's? Let $m_1$ be the number of interceptors attacking the first ICBM, let $m_2$ be the number of interceptors attacking the second ICBM, and finally let $m_n$ be the number of interceptors attacking the $n^{th}$ ICBM. It is obvious that the number of different possibilities of attributing $m_1$, $m_2$, ... $m_n$ interceptors to the first, second, ... $n^{th}$ ICBM is

$$\frac{m!}{m_1!\, m_2! \, \ldots \, m_n!}$$

---

Released by author, September 1961

Since $n^m$ is the total number of possible distributions of m "elements" among n "boxes," the distribution $m_1, m_2, \ldots m_n$ has the probability

$$\frac{1}{n^m} \cdot \frac{m!}{m_1!\, m_2! \ldots m_n!}$$

The probabilities that the first, second, ... $j^{th}$ ICBM's escape destruction are $q^{m_1}, q^{m_2}, \ldots q^{m_j}$, respectively, where $q = 1 - p$.

The probabilities that the $(j+1)^{st}, \ldots n^{th}$ ICBM's are destroyed are $1 - q^{m_{j+1}}, \ldots 1 - q^{m_n}$, respectively.

Hence the configuration: first j ICBM's not destroyed, remaining n - j ICBM's destroyed, $i^{th}$ ICBM attacked by $m_i$ interceptors $(i = 1, \ldots n)$, has the probability

$$\frac{1}{n^m} \cdot \frac{m!}{m_1!\, m_2! \ldots m_n!} q^{m_1} \ldots q^{m_j} \left(1 - q^{m_{j+1}}\right) \ldots \left(1 - q^{m_n}\right)$$

Since there are $\binom{n}{j}$ possibilities of selecting a set of j escapes out of the n ICBM's we obtain

$$P_j = \binom{n}{j} \frac{1}{n^m} \sum \frac{m!}{m_1!\, m_2! \ldots m_n!} q^{m_1} \ldots q^{m_j} \left(1 - q^{m_{j+1}}\right) \ldots \left(1 - q^{m_n}\right) \tag{1}$$

where $P_j$ is the probability that exactly j ICBM's escape destruction

and where the sum has to be extended over all configurations $m_1, m_2, \dots m_n$ so that

$$m_1 + m_2 + \cdots m_n = m$$

The sum in formula (1) may be written in a more convenient form. We substitute in equation (1)

$$m_1 + m_2 + \cdots m_j = \nu$$

Since each term of the sum in equation (1) for which $m_{j+k} = 0$ ($k \geq 1$) vanishes, we have to consider only those configurations for which

$$m_{j+1} + \cdots m_n = m - (m_1 + \cdots m_j) = m - \nu \geq n - j$$

or

$$\nu \leq m - n + j$$

The set of the $\nu$ elements $m_1, m_2, \dots m_j$ can be selected in $\binom{m}{\nu}$ ways and there are $j^\nu$ possibilities of attributing $\nu$ interceptors to j ICBM's. The number of possibilities of attributing the remaining $m - \nu$ interceptors to the remaining $n - j$ ICBM's is

$$\frac{(m - \nu)!}{m_{j+1}! \cdots m_n!}$$

where

$$m_{j+1} + \cdots m_n = m - \nu$$

Hence it follows

$$P_j = \binom{n}{j} \frac{1}{n^m} \sum_{\nu=0}^{m-n+j} \binom{m}{\nu} \left(jq\right)^{\nu} \sum \frac{(m - \nu)!}{(m_{j+1})! \ \ldots \ m_n!} \left(1 - q^{m_{j+1}}\right) \cdots \left(1 - q^{m_n}\right) \tag{2}$$

where the second sum has to be extended over all possible configurations $m_{j+1} \ \ldots \ m_n$ satisfying the condition

$$m_{j+1} + \cdots m_n = m - \nu \tag{3}$$

After determining $P_j$ by means of formula (2) we see that the probability $p_\mu$ of killing at least $n - \mu$ ICBM's is

$$p_\mu = \sum_0^\mu P_j$$

In order to estimate the sum

$$S = \sum \frac{(m - \nu)!}{(m_{j+1})! \ \ldots \ m_n!} \left(1 - q^{m_{j+1}}\right) \ldots \left(1 - q^{m_n}\right) \tag{4}$$

the following study is useful. In the sum (4) all terms vanish if

any $m_{j+k} = 0$ $(k \geq 1)$. If we now replace $\left(1 - q^{m_{j+1}}\right) \cdots \left(1 - q^{m_n}\right)$ by its maximum $M_\nu$ under the boundary condition (3) we see that

$$S \leq M_\nu \cdot \sum \frac{(m - \nu)!}{(m_{j+1})! \ \cdots \ m_n!} \tag{5}$$

where only those configurations have to be considered for which

$$m_{j+k} \geq 1$$

This leads to the following problem: How many possibilities are there to distribute $\rho$ "elements" to $\sigma$ "boxes" $(\rho \geq \sigma)$ so that each box contains at least one of the $\rho$ elements?

In order to answer this question we shall derive a recursion formula as follows: Let $\phi_\rho(\sigma, j)$ be the number of possible distributions so that the first $j$ boxes contain at least one element. It is evident that $\phi_\rho(\sigma, j+1)$ equals $\phi_\rho(\sigma, j)$ minus that number of distributions for which the first $j$ boxes contain at least one element and for which the $(j+1)^{st}$ box is empty. This number is $\phi_\rho(\sigma-1, j)$. Therefore:

$$\phi_\rho(\sigma, j+1) = \phi_\rho(\sigma, j) - \phi_\rho(\sigma-1, j) \tag{6}$$

We now have

$$\phi_\rho(\sigma, 0) = \sigma^\rho \tag{7}$$

From equations (6) and (7) we conclude by induction

$$\Phi_\rho(\sigma, \sigma) = \sigma^\rho - \sigma(\sigma-1)^\rho + \binom{\sigma}{2}(\sigma-2)^\rho - \cdots = \sum_{\nu=0}^{\sigma} (-1)^\nu \binom{\sigma}{\nu} (\sigma-\nu)^\rho \tag{8}$$

or

$$\Phi_\rho(\sigma, \sigma) = \sum_{\nu=0}^{\sigma} (-1)^\nu \binom{\sigma}{\nu} \nu^\rho \tag{9}$$

where $\Phi_\rho(\sigma, \sigma)$ means the number of possibilities to distribute $\rho$ "elements" to $\sigma$ "boxes" so that each box contains at least one element. It is desirable to express $\Phi_\rho(\sigma, \sigma)$ in a closed form. For this purpose we write

$$\sum_{\nu=0}^{\sigma} (-1)^\nu \binom{\sigma}{\nu} e^{\nu x} = (e^x - 1)^\sigma \tag{10}$$

Differentiating (10) $\rho$ times with respect to x we obtain

$$\Phi_\rho(\sigma, \sigma) = \left.\frac{d^\rho}{dx^\rho}(e^x - 1)^\sigma\right|_{x=0} \tag{11}$$

which yields $\Phi_\rho(\sigma, \sigma)$ in the desired closed form.

Equation (11) shows that $\Phi_\rho(\sigma, \sigma)$ may be written as a contour integral in the Gaussian plane, extended over any closed contour encircling the origin once

$$\Phi_\rho(\sigma, \sigma) = \frac{\rho!}{2\pi i} \oint \frac{(e^z - 1)^\sigma}{z^{\rho+1}} dz \tag{12}$$

In various ways we can obtain useful estimates for $\Phi_\rho(\sigma, \sigma)$, for instance choosing the unit circle about the origin as contour. However, we shall not dwell upon this method any longer, because the fundamental formula (2) can be transformed into an interesting and much simpler form, handy above all for computations with a digital computer.

With regard to practical calculations the greatest difficulty in evaluating formula (2) is caused by the factorial terms and, above all, by decomposing m into a large number of summands. How this can be avoided shall now be investigated.

The second sum on the right-hand side of equation (2) has the structure

$$\sum \frac{s!}{s_1! \cdots s_t!} \left(1 - q^{s_1}\right) \cdots \left(1 - q^{s_t}\right) \tag{13}$$

where

$$s_1 + \cdots s_t = s$$

Eliminating the parentheses in (13) we have $\binom{t}{\alpha}$ products of the kind $q^{s_1} q^{s_2} \dots q^{s_\alpha}$. All these $\binom{t}{\alpha}$ products contribute the same value

$$S = (-1)^\alpha \sum \frac{s!}{s_1! \dots s_t!} q^{s_1} \dots q^{s_\alpha} \tag{14}$$

to the sum in expression (13) if we first keep $s_1, s_2, \dots s_\alpha$ fixed. We substitute $s_1 + s_2 + \dots s_\alpha = \tilde{s}$ and write (14) in the form

$$S = (-1)^\alpha \sum \left( \frac{\frac{s!}{(s - \tilde{s})!} \cdot q^{\tilde{s}}}{s_1! \, s_2! \dots s_\alpha!} \right) \left( \frac{(s - \tilde{s})!}{s_{\alpha+1}! \dots s_t!} \right) \tag{15}$$

Keeping first $s_1, s_2, \dots s_\alpha$ fixed and summing over $s_{\alpha+1}, \dots s_t$, we obtain

$$S = \sum_{\tilde{s}=0}^{s} (-1)^\alpha (t - \alpha)^{s-\tilde{s}} q^{\tilde{s}} \binom{s}{\tilde{s}} \sum \frac{\tilde{s}!}{s_1! \dots s_\alpha!} \tag{16}$$

Summing over $s_1, \ldots s_\alpha$, and keeping $\tilde{s}$ fixed during this summation, we obtain

$$S = \sum_{\tilde{s}=0}^{s} (-1)^\alpha (t - \alpha)^{s-\tilde{s}} q^{\tilde{s}} \binom{s}{\tilde{s}} \alpha^{\tilde{s}} \tag{17}$$

or

$$S = (-1)^\alpha (t - \alpha p)^s \tag{18}$$

Equation (18) now leads to the important result

$$\begin{aligned} \sum \frac{s!}{s_1! \ldots s_t!} \left(1 - q^{s_1}\right) \cdots \left(1 - q^{s_t}\right) \\ = \sum_{\alpha=0}^{t} (-1)^\alpha \binom{t}{\alpha} (t - \alpha p)^s \end{aligned} \tag{19}$$

Equation (19) finally enables us to give an expression for $P_j$ much more suitable for practical calculations:

$$P_j = \binom{n}{j} \frac{1}{n^m} \sum_{\nu=0}^{m-n+j} \binom{m}{\nu} \left(j q\right)^\nu \sum_{\alpha=0}^{n-j} (-1)^\alpha \binom{n-j}{\alpha} \left(n - j - \alpha p\right)^{m-\nu} \tag{20}$$

Let us now represent $P_j$ as a contour integral. For this purpose we apply to the second sum in equation (20) a method analogous to that which led to equation (11). From the relation

$$\sum_{\alpha=0}^{n-j} (-1)^{\alpha} \binom{n-j}{\alpha} e^{(n-j-\alpha p)x} = e^{(n-j)x} \left(1 - e^{-px}\right)^{n-j} \tag{21}$$

we conclude by differentiating $(m - \nu)$ times with respect to $x$, and subsequently substituting $x = 0$

$$\sum_{\alpha=0}^{n-j} (-1)^{\alpha} \binom{n-j}{\alpha} \left(n-j-\alpha p\right)^{m-\nu} = \frac{d^{m-\nu}}{dx^{m-\nu}} \left\{ e^{(n-j)x} \left(1 - e^{-px}\right)^{n-j} \right\} \Bigg|_{x=0} \tag{22}$$

or

$$\sum_{\alpha=0}^{n-j} (-1)^{\alpha} \binom{n-j}{\alpha} \left(n-j-\alpha p\right)^{m-\nu} = \frac{(m-\nu)!}{2i\pi} \oint \frac{e^{(n-j)z}\left(1 - e^{-pz}\right)^{n-j}}{z^{m-\nu+1}} \, dz \tag{23}$$

where the contour integral has to be extended over a closed contour about the origin in the Gaussian plane of the complex variable $z$.

Substituting equation (23) in equation (20) we obtain after some elementary operations

$$P_j = \binom{n}{j} \cdot \frac{m!}{n^m} \cdot \frac{1}{2i\pi} \oint \sum_{\nu=0}^{m-n+j} \frac{(jqz)^\nu}{\nu!} \frac{\left(1 - e^{-pz}\right)^{n-j} e^{(n-j)z}}{z^{m+1}} dz \tag{24}$$

The sum

$$\sum_{\nu=0}^{m-n+j} \frac{(jqz)^\nu}{\nu!} \tag{25}$$

can rigorously be replaced by $e^{jqz}$ because powers of jqz higher than m-n+j would not yield any residue different from zero in the integral of equation (24). Hence it follows that $P_j$ can be represented in the form

$$P_j = \binom{n}{j} \cdot \frac{1}{n^m} \cdot \frac{m!}{2i\pi} \oint \frac{e^{(n-jp)z} \left(1 - e^{-pz}\right)^{n-j}}{z^{m+1}} dz \tag{26}$$

It is appealing to check equation (26) by proving that indeed

$$\sum_{j=0}^{n} P_j = 1 \tag{27}$$

Since

$$\sum_{j=0}^{n} \binom{n}{j} e^{-jpz} \left(1 - e^{-pz}\right)^{n-j} = 1$$

we obtain from equation (26)

$$\sum_{j=0}^{n} P_j = \frac{m!}{n^m} \cdot \frac{1}{2i\pi} \oint \frac{e^{nz}}{z^{m+1}} \, dz \tag{28}$$

Since

$$\oint \frac{e^{nz}}{z^{m+1}} \, dz = 2i\pi \frac{n^m}{m!}$$

equation (27) has been verified.

Equation (26) can be written in the form

$$P_j = \binom{n}{j} \cdot \frac{1}{n^m} \cdot \frac{m!}{2i\pi} \oint \frac{\left(e^z - e^{qz}\right)^n \left(e^{pz} - 1\right)^{-j}}{z^{m+1}} \, dz \tag{29}$$

It suggests eliminating the parentheses in equation (26) in order to calculate the residues. After some elementary transformations we obtain:

$$P_j = \left(1 - \frac{jp}{n}\right)^m \binom{n}{j} \sum_{\mu=0}^{n-j} (-1)^\mu \binom{n-j}{\mu} \left(1 - \frac{\mu p}{n-jp}\right)^m \tag{30}$$

For practical computations it is always advisable to comprehend two succeeding terms in the sum of equation (30):

$$\binom{n-j}{\mu}\left(1 - \frac{\mu p}{n-jp}\right)^m - \binom{n-j}{\mu+1}\left(1 - \frac{(\mu+1)p}{n-jp}\right)^m$$

$$= \binom{n-j}{\mu}\left(1 - \frac{\mu p}{n-jp}\right)^m \left\{1 - \frac{n-j-\mu}{\mu+1}\left(1 - \frac{p}{n-jp-\mu p}\right)^m\right\} \tag{31}$$

$$\approx \binom{n-j}{\mu}\left(1 - \frac{\mu p}{n-jp}\right)^m \left(1 - \frac{n-j-\mu}{\mu+1}\, e^{-\frac{mp}{n-jp-\mu p}}\right)$$

This approximation is sufficient for all practical purposes. Finally equation (30) assumes the form:

$$P_j \approx \binom{n}{j}\left(1 - \frac{jp}{n}\right)^m \sum_{\sigma=0}^{\frac{n-j-1}{2}} \binom{n-j}{2\sigma}\left(1 - \frac{2\sigma p}{n-jp}\right)^m \left(1 - \frac{n-j-2\sigma}{2\sigma+1}\, e^{-\frac{mp}{n-jp-2\sigma p}}\right) \tag{32}$$

for an odd (n-j). For an even (n-j) the upper limit of the sum equals

$$\frac{(n-j)}{2}$$

Professor Harry Carver suggested that we calculate the factorial moments of our distribution function, remarking that the general expressions would probably become very simple.

The $\nu^{th}$ factorial moment $\mu_{(\nu)}$ is defined as follows:

$$\mu_{(\nu)} = \sum_{j=0}^{n} j(j-1)\ldots(j-\nu+1)\, P_j \tag{33}$$

or using the notation:

$$j(j-1)\ldots(j-\nu+1) = j^{(\nu)}$$

we have:

$$\mu_{(\nu)} = \sum_{j=0}^{n} j^{(\nu)}\, P_j \tag{34}$$

In order to calculate these moments we make use of formula (26) which we shall write in a slightly modified form:

$$P_j = \binom{n}{j} \cdot \frac{1}{n^m} \cdot \frac{m!}{2i\pi} \oint \frac{e^{nqz}\left(e^{pz} - 1\right)^{n-j}}{z^{m+1}}\, dz \tag{35}$$

From equation (35) we conclude:

$$\mu(\nu) = \frac{m!}{2i\pi\, n^m} \oint \frac{e^{nqz}\left(e^{pz} - 1\right)^n}{z^{m+1}} \sum_{j=0}^{n} j^{(\nu)} \binom{n}{j} \left(e^{pz} - 1\right)^{-j} dz \tag{36}$$

The sum in the integrand can easily be evaluated as follows: We substitute,

$$e^{pz} - 1 = x^{-1}$$

Since:

$$\sum_{j=0}^{n} \binom{n}{j} \left(e^{pz} - 1\right)^{-j} = \sum_{j=0}^{n} \binom{n}{j} x^j = \left(1 + x\right)^n \tag{37}$$

we obtain by deriving equation (37) $\nu$ times with respect to x and multiplying the result by $x^\nu$:

$$\sum_{j=0}^{n} j^{(\nu)} \binom{n}{j} x^j = n^{(\nu)}\, x^\nu \left(1 + x\right)^{n-\nu} \tag{38}$$

or

$$\sum_{j=0}^{n} j^{(\nu)} \binom{n}{j} \left(e^{pz} - 1\right)^{-j} = n^{(\nu)}\, e^{pz(n-\nu)} \left(e^{pz} - 1\right)^{-n} \tag{39}$$

We now substitute equation (39) in equation (36):

$$\mu_{(\nu)} = \frac{m! \cdot n^{(\nu)}}{2i\pi\, n^m} \oint \frac{e^{(n - \nu p)z}}{z^{m+1}}\, dz \tag{40}$$

Calculating the residue of the integrand of equation (40) we finally obtain:

$$\boxed{\mu_{(\nu)} = n^{(\nu)} \left( 1 - \frac{\nu p}{n} \right)^m} \tag{41}$$

Our next aim is to give an approximation of our distribution function good to any desired degree of accuracy, taking advantage of the simple expressions for the factorial moments.

We are particularly interested in the case where n is very large i.e., $n \geqq 200$. In this case it is justified to replace the summation sign in equation (34) by an integral sign:

$$\mu_{(\nu)} = \int_{j=0}^{n} j^{(\nu)}\, P_{(j)}\, d_j \tag{42}$$

We now perform the similarity transformation:

$$j = \frac{n}{2}\, (x + 1) \tag{43}$$

which transforms the interval $(0, n)$ into the interval $(-1, 1)$, and obtain:

$$\mu_{(\nu)} = \left(\frac{n}{2}\right)^{\nu} \int_{-1}^{1} \left(1 + x\right)\left(1 + x - \frac{2}{n}\right)\left(1 + x - \frac{4}{n}\right) \dots \left(1 + x - \frac{2\nu-2}{n}\right) \tilde{P}(x)\, dx \tag{44}$$

where $\tilde{P}(x)\, dx$ is the probability that $x$ lies in the interval $(x, x + dx)$.

Now we take advantage of the simple expressions for the factorial moments in order to expand $\tilde{P}(x)$ into a series of Legendre polynomials. For this purpose we eliminate the parentheses in equation (44):

$$\mu_{(\nu)} = \left(\frac{n}{2}\right)^{\nu} \left[ \int_{-1}^{1} \left(1 + x\right)^{\nu} \tilde{P}(x)\, dx - \frac{2}{n}\, s_1^{(\nu-1)} \int_{-1}^{1} \left(1 + x\right)^{\nu-1} \tilde{P}(x)\, dx \right. \tag{45}$$

$$\left. + \frac{2^2}{n^2}\, s_2^{(\nu-1)} \int_{-1}^{1} \left(1 + x\right)^{\nu-2} \tilde{P}(x)\, dx \dots + \frac{(-2)^{\nu-1}}{n^{\nu-1}} \right]$$

where $s_i^{(\nu-1)}$ means the $i^{\text{th}}$ elementary symmetric function of the first $\nu - 1$ integers.

For practical applications we shall list the first four symmetric functions $s_1^{(n)}$, $s_2^{(n)}$, $s_3^{(n)}$, $s_4^{(n)}$ :

$$s_1^{(n)} = \frac{n(n+1)}{2}$$

$$s_2^{(n)} = \frac{n(n+1)(n-1)(3n+2)}{24}$$

$$s_3^{(n)} = \frac{n^2(n+1)^2(n-2)(n-1)}{48} \tag{46}$$

$$s_4^{(n)} = \frac{n(n+1)(n-1)(n-2)(n-3)(15n^3+15n^2-10n-8)}{2^7 \cdot 3^2 \cdot 5}$$

Equation (45) suggests expanding the expression $(1+x)^n$ into a sum of Legendre polynomials. As will be shown in the Appendix, we find:

$$(1+x)^n = \sum_{i=0}^{n} I_{in}\,\overline{P}_i \tag{47}$$

where $\overline{P}_i$ is the $i^{th}$ Legendre polynomial normalized so that:

$$\int_{-1}^{1} \overline{P}_i^2\,dx = 1 \tag{48}$$

and

$$I_{in} = \sqrt{\frac{2i + 1}{2}}$$

$$\frac{2^{n+1} \, (n!)^2}{(2i + 1)! \left[(n+1)n - (i+1)i\right] \left[n(n-1) - (i+1)i\right] \ldots \left[(i+2)\,(i+1) - (i+1)\,i\right]} \tag{49}$$

Let us now expand $\tilde{P}(x)$ into a series of Legendre polynomials:

$$\tilde{P}(x) = \sum_{0}^{\infty} a_j \, \bar{P}_j \, (x) \tag{50}$$

We now substitute equations (41), (47), and (50) in equation (45):

$$n^{(\nu)} \left(1 - \frac{\nu p}{n}\right)^m = \left(\frac{n}{2}\right)^{\nu} \sum_{i=0}^{\nu} I_{i\nu} \, a_i - \frac{2}{n} \, s_1^{(\nu-1)} \sum_{i=0}^{\nu-1} I_{i\nu-1} \, a_i$$

$$+ \frac{2^2}{n^2} \, s_2^{(\nu-1)} \sum_{i=0}^{\nu-2} I_{i\nu-2} \, a_i \ldots \tag{51}$$

Equation (51) enables us to calculate easily the coefficients $a_i$ successively, and is above all suitable for large numbers n and m.

Although equation (51) apparently is complicated, it permits computation of the coefficients $a_i$ without solving a system of linear equations. The numerical evaluation of equation (51) can be performed by means of a desk calculator.

## APPENDIX

In order to prove formula (49), page 19, we start from the differential equation for Legendre's polynomials:

$$\frac{d}{dx}\left[(x^2 - 1)\, P_i'\right] = i(i + 1)\, P_i \tag{1}$$

We now multiply equation (1) by $(1 + x)^n$ and integrate from $-1$ to $+1$ :

$$i(i + 1) \int_{-1}^{+1} (1 + x)^n\, P_i\, dx = \int_{-1}^{1} \frac{d}{dx}\left[(x^2 - 1)\, P_i'\right] (1 + x)^n\, dx$$

$$= - n \int_{-1}^{1} (x^2 - 1)\, (1 + x)^{n-1}\, P_i'\, dx = - n \int_{-1}^{1} (x - 1)\, (1 + x)^n\, P_i'\, dx$$

$$= n \int_{-1}^{1} (1 + x)^n\, P_i\, dx + n^2 \int_{-1}^{1} (x - 1)\, (1 + x)^{n-1}\, P_i\, dx \tag{2}$$

$$= (n + n^2) \int_{-1}^{1} (1 + x)^n\, P_i\, dx - 2n^2 \int_{-1}^{1} (1 + x)^{n-1}\, P_i\, dx$$

From equation (2) we derive the recursion formula:

$$I_{in} = \frac{2n^2}{n^2 + n - i^2 + i}\, I_{in-1} \tag{3}$$

We apply formula (3) successively until we reach $I_{ii}$, which is:

$$I_{ii} = \int_{-1}^{1} (1 + x)^{i} P_{i} \, dx = \int_{-1}^{1} x^{i} P_{i} \, dx \qquad (4)$$

The integral (4) is well known and may be taken from any textbook*, and from it we then obtain, after some elementary transformations, equation (49), page 19.

*W. Magnus and F. Oberhettinger, Formeln und Sätze für die speziellen Funktionen der mathematischen Physik.

# DISTRIBUTION

| | | | |
|---|---|---|---|
| AFOSR (SRGL)<br>Wash 25, DC | 2 | Ames Research Center (NASA)<br>ATTN: Technical Library<br>Moffett Field, Calif | 1 |
| NASA (Library)<br>1520 H Street NW<br>Wash 25, DC | 2 | RAND Corporation<br>1700 Main Street<br>Santa Monica, Calif | 2 |
| Chief, R&D, Dept of the Army<br>ATTN: Scientific Information Branch<br>Wash 25, DC | 1 | ASTIA (TIPCR)<br>Arlington Hall Station<br>Arlington 12, Va | 20 |
| AFSC (SCRS)<br>Andrews AFB<br>Wash 25, DC | 1 | Langley Research Center (NASA)<br>ATTN: Technical Library<br>Langley AFB, Va | 1 |
| Chairman<br>Canadian Joint Staff (DRB/DSIS)<br>2450 Massachusetts Avenue NW<br>Wash 25, DC | 1 | Lewis Research Center (NASA)<br>ATTN: Technical Library<br>21000 Brookpark Road<br>Cleveland 35, Ohio | 1 |
| U. S. Naval Research Laboratory<br>ATTN: Library<br>Wash 25, DC | 1 | ARGMA (ORDXR-OTL)<br>Redstone Arsenal, Ala | 1 |
| Institute of Technology (AU) Library<br>MCLI-LIB, Bldg 125, Area B<br>Wright-Patterson AFB, Ohio | 1 | Institute of Aeronautical Sciences<br>2 East 64th Street<br>New York 21, NY | 1 |
| ASD (Technical Library)<br>Wright-Patterson AFB, Ohio | 1 | Applied Mechanics Reviews<br>Southwest Research Institute<br>8500 Culebra Road<br>San Antonio 6, Tex | 2 |
| ARL (Technical Library)<br>Bldg 450<br>Wright-Patterson AFB, Ohio | 2 | AFCRL (CRRELA)<br>L. G. Hanscom Field, Mass | 1 |
| High Speed Flight Station (NASA)<br>ATTN: Technical Library<br>Edwards AFB, Calif | 1 | AEDC (AEOIM)<br>Arnold AF Stn, Tenn | 1 |
| AFFTC (FTOTL)<br>Edwards AFB, Calif | 1 | Detachment 1<br>Hq, Office of Aerospace Research<br>European Office, USAF<br>47 Cantersteen<br>Brussels, Belgium | 1 |

Signal Corps Engineering Laboratory 1
(SIGFM/EL-RPO)
Fort Monmouth, NJ

Linda Hall Library 1
ATTN: Documents Division
5109 Cherry Street
Kansas City 10, Mo

HQ USAF (AFCIN-3T) 1
Wash 25, DC

HQ USAF (AFDRD-HF) 1
Wash 25, DC

CIA (OCR Mail Room) 2
2430 E Street NW
Wash 25, DC

Dept of the Navy 1
Bureau of Ordnance (Sp-401)
Wash 25, DC

AFMTC (Tech Library MU-135) 1
Patrick AFB, Fla

APGC (PGTRIL) 1
Eglin AFB, Fla

AFSWC (SWOI) 1
Kirtland AFB, NMex

AU (AUL-6008) 1
Maxwell AFB, Ala

RADC (RCOIL-2) 1
Griffiss AFB, NY

Dept of the Navy 1
Naval Research Laboratory
ATTN: Director, Code 5360
Wash 25, DC

Commanding Officer 1
Diamond Ordnance Fuse Laboratories
ATTN: Technical Reference Section
(ORDTL 06.33)
Wash 25, DC

USAFA (DLIB) 2
U.S. Air Force Academy
Colo

Analytic Services, Inc. 1
1101 North Royal Street
Alexandria, Va

School of Aeronautical and 1
Engineering Sciences
Perdue University
ATTN: Aero and ES Library
Lafayette, Ind

Space Technology Laboratories, 1
Inc. ATTN: Information Services
Document Acquisition Group
P. O. Box 95001
Los Angeles 45, Calif

Commanding General 1
ATTN: ORDBS-OM-TL 312
White Sands Missile Range
NMex

Lt Col K. B. Dobson 1
Ordnance Mission
British Liaison Office
White Sands Missile Range
NMex

New Mexico State University of 1
Agriculture, Engineering, and
Science
ATTN: Library
University Park, NMex

University of New Mexico 1
Government Publications Division
University of New Mexico Library
Albuquerque, NMex

LOCAL

| | |
|---|---|
| NLO | 1 |
| SRL | 1 |
| SRLT | 15 |
| SRLTL | 3 |

Directorate of Research Analysis
Air Force Office of Scientific Research
Office of Aerospace Research
Holloman AFB, New Mexico

PROBABILITY CONSIDERATIONS OF DESTROYING ICBM's WITH INTERCEPTOR SATELLITES, by Herbert Knothe, November 1961. 23 pp. (AFOSR/DRA-61-3) Unclassified Report

This paper presents a method for calculating the probability that a certain minimum number of a set of ICBM's can be destroyed by a given number of interceptor satellites.

UNCLASSIFIED

UNCLASSIFIED

Directorate of Research Analysis
Air Force Office of Scientific Research
Office of Aerospace Research
Holloman AFB, New Mexico

PROBABILITY CONSIDERATIONS OF DESTROYING ICBM's WITH INTERCEPTOR SATELLITES, by Herbert Knothe, November 1961. 23 pp. (AFOSR/DRA-61-3) Unclassified Report

This paper presents a method for calculating the probability that a certain minimum number of a set of ICBM's can be destroyed by a given number of interceptor satellites.

UNCLASSIFIED

UNCLASSIFIED

Directorate of Research Analysis
Air Force Office of Scientific Research
Office of Aerospace Research
Holloman AFB, New Mexico

PROBABILITY CONSIDERATIONS OF DESTROYING ICBM's WITH INTERCEPTOR SATELLITES, by Herbert Knothe, November 1961. 23 pp. (AFOSR/DRA-61-3) Unclassified Report

This paper presents a method for calculating the probability that a certain minimum number of a set of ICBM's can be destroyed by a given number of interceptor satellites.

UNCLASSIFIED

UNCLASSIFIED

Directorate of Research Analysis
Air Force Office of Scientific Research
Office of Aerospace Research
Holloman AFB, New Mexico

PROBABILITY CONSIDERATIONS OF DESTROYING ICBM's WITH INTERCEPTOR SATELLITES, by Herbert Knothe, November 1961. 23 pp. (AFOSR/DRA-61-3) Unclassified Report

This paper presents a method for calculating the probability that a certain minimum number of a set of ICBM's can be destroyed by a given number of interceptor satellites.

UNCLASSIFIED

UNCLASSIFIED